数理解析研究所講究録
203巻 1974年 59-71

FINITE GROUPS WHICH ACT FREELY ON SPHERES

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We will study the problem: Let G be a finite group which acts freely (and topologically) on the sphere S^{2t-1}. Can G act freely and orthogonally on S^{2t-1}?

The result of T. Petrie [5] shows that the answer is no for t odd prime. The problem for $t = 2$ is unsolved at present (see [2],[3],[4]). In this note it will be shown that the answer is yes for $t = 4$, and also for $t = 2^{\nu}$ ($\nu \geqq 3$) if G is solvable.

1. Preliminary theorems

By J. Milnor [3] we have

(1.1) If G is a group which acts freely on S^n, then G satisfies the following conditions which are equivalent:

i) Any element of order 2 in G belongs to the center of G.

ii) G has at most one element of order 2.

The following (1.2) and (1.3) are shown in [1].

(1.2) If G acts freely on S^n, the cohomology of G has period $n + 1$.

(1.3) The following two conditions are equivalent:

i) A finite group G has periodic cohomology.

ii) Every abelian subgroup of G is cyclic.

A complete classification of finite groups satisfying the condition ii) of (1.3) is known by H. Zassenhaus [11] and M. Suzuki [6].

For future reference we reproduce it below after J. Wolf [10] and C.B. Thomas-C.T.C. Wall [8].

(1.4) Let G be a finite group satisfying the condition ii) of (1.3). If G is solvable, it is one of the following groups:

Type	Generators	Relations	conditions	order
I	A, B	$A^m=B^n=1$, $BAB^{-1}=A^r$	$m \geqq 1$, $n \geqq 1$, $(n(r-1), m)=1$, $r^n \equiv 1$ (m)	mn
II	A, B, R	As in I ; also $R^2 = B^{n/2}$, $RAR^{-1}=A^{\ell}$, $RBR^{-1}=B^k$	As in I ; also $\ell^2 \equiv r^{k-1} \equiv 1$ (m), $n=2^u v$, $u \geqq 2$, $k \equiv -1$ (2^u), $k^2 \equiv 1$ (n)	2mn
III	A, B, P, Q	As in I ; also $P^4=1$, $P^2=Q^2=(PQ)^2$, $AP=PA$, $AQ=QA$, $BPB^{-1}=Q$, $BQB^{-1}=PQ$	As in I ; also $n \equiv 1$ (2), $n \equiv 0$ (3)	8mn
IV	A, B, P, Q, R	As in III ; also $R^2=P^2$, $RPR^{-1}=QP$ $RQR^{-1} = Q^{-1}$, $RAR^{-1}=A^{\ell}$, $RBR^{-1}=B^k$	As in III ; also $k^2 \equiv 1$ (n), $k \equiv -1$ (3), $r^{k-1} \equiv \ell^2 \equiv 1$ (m)	16mn

If G is non-solvable, it is one of the following groups.

V. $G = K \times SL(2, p)$, where p is a prime $\geqq 5$, and K is a group of type I and order prime to $|SL(2, p)| = p(p^2 - 1)$.

VI. G is generated by a group of type V and an element S

<u>such that</u>

$$\underline{S^2 = -1 \in SL(2,\ p), \quad SAS^{-1} = A^{-1}},$$

$$\underline{SBS^{-1} \doteq B, \quad SLS^{-1} = \theta(L) \quad (L \in SL(2,\ p))}.$$

Here, $SL(2,\ p)$ denotes the multiplicative group of 2×2 matrices of determinant 1 with entries in the field Z_p, and θ is an automorphism of $SL(2,\ p)$ given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -\omega, & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1, & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix},$$

ω being a generator of the multiplicative group in Z_p.

Let G be any finite group, and p a prime. Then the <u>p-period</u> of G is defined to be the least positive integer q such that the Tate cohomology groups $\hat{H}^i(G\ ;\ A)$ and $\hat{H}^{i+q}(G\ ;\ A)$ have isomorphic p-primary components for all i and all A. The period of G is the least common multiple of all the p-periods. R.G. Swan [7] gave a method to calculate the p-period as follows:

(1.5) (i) <u>If a 2-Sylow subgroup of a finite group G is cyclic, the 2-period of G is 2. If a 2-Sylow subgroup of G is a generalized quaternion group, the 2-peirod of G is 4</u>.

(ii) <u>Suppose p is odd and a p-Sylow subgroup G_p of G is cyclic. Let Φ_p denote the group of automorphisms of G_p induced by inner automorphisms of G. Then the p-period of G is $2|\Phi_p|$</u>.

If $N(G_p)$, $C(G_p)$ denote the normalizer and centralizer of

G_p, it holds $\Phi_p \cong N(G_p)/C(G_p)$. From this we have the following (see [8]).

(1.6) <u>If a 3-Sylow subgroup of G is cyclic, the 3-period of G divides 4.</u>

We shall next consider free orthogonal actions on S^n. If a representation ρ of a group G is said to be <u>fixed point free</u> if $1 \neq g \in G$ implies that $\rho(g)$ does not have $+1$ for an eigenvalue.

With the notations of (1.4), let d denote the order of r in the multiplicative group of residues modulo m of integers prime to m. Modifying the work of G. Vincent [9], J.Wolf proves the following (1.7),(1.8) in [10].

(1.7) <u>For a finite group G, the following two conditions are equivalent:</u>

i) <u>G has a fixed point free complex representation.</u>

ii) <u>G is of type I, II, III, IV, V for $q = 5$, or VI for $q = 5$, with the additional condition: n/d is divisible by every prime divisor of d.</u>

(1.8) <u>Let G be a finite group satisfying the conditions in (1.7). Then each fixed point free, irreducible complex representation of G has the degree $\delta(G)$ which is given as follows:</u>

Type	I	II	III	IV'	IV"	V	VI
$\delta(G)$	d	2d	2d	2d	4d	2d	4d

<u>If $|G| > 2$, G acts freely and orthogonally on S^{2q-1} if and</u>

<u>only if q is divisible by $\delta(G)$.</u>

Here IV' refers to G of type IV such that $G = \{A, B^3\} \times O^*$ and $|G| \not\equiv 0 \ (9)$, O^* being the binary octahedral group ; IV" refers to G of type IV which is not of type IV'.

2. <u>Finite groups acting freely on $S^{2^\nu-1}$</u>

We shall consider the following conditions for a finite group G:

(A_ν) <u>G can act freely and orthogonally on $S^{2^\nu-1}$.</u>

(B_ν) <u>G can act freely on $S^{2^\nu-1}$.</u>

(C_ν) <u>G has the cohomology of period 2^ν and has at most one element of order 2.</u>

$(A_\nu) \Rightarrow (B_\nu)$ is trivial, and $(B_\nu) \Rightarrow (C_\nu)$ holds by (1.2) and (1.3). We shall study whether $(C_\nu) \Rightarrow (A_\nu)$ holds.

Let G be a finite group satisfying (C_ν). Then, by (1.3) and (1.4), G is of type I, II, III, IV, V or VI. We shall retain the notations in § 1.

Case 1: $m \neq 1$.

Since it follows from the conditions of type I that m is odd, there is an odd prime p such that $m = p^c m'$, $(m', p) = 1$. Put $A' = A^{m'}$, then A' generates a cyclic group of order p^c. If we observe the order of G, it follows that this cyclic group is a p-Sylow subgroup of G. Since

$$B^i A' B^{-i} = A'^{r^i} \quad (i = 0, 1, \cdots, d-1)$$

are distinct, it follows from (1.5) that the period of G is a multiple of $2d$. Therefore 2^{ν} is a multiple of $2d$, and so d is a divisor of $2^{\nu-1}$. Since $m = 1$ is equivalent to $d = 1$, we have

$$d = 2^{\alpha} \quad \text{with} \quad \alpha = 1,2,\cdots, \nu - 1.$$

Since n is a multiple of d, n is even. Therefore G can not be of type III, IV, V or VI. If G is of type II and $d = 2^{\alpha}$ with $\alpha \geqq 2$, the conditions on k yield a contradiction. Thus G is of type I with $d = 2^{\alpha}$ $(\alpha = 1,2,\cdots, \nu - 1)$, or of type II with $d = 2$.

Since the order of $B^{n/2}$ is 2, by (1.1) we have

$$B^{n/2}AB^{-n/2} = A.$$

Since $BAB^{-1} = A^{r}$, we have also

$$B^{n/2}AB^{-n/2} = A^{r^{n/2}}.$$

Hence $r^{n/2} \equiv 1\ (m)$, and $n/2$ is a multiple of $d = 2^{\alpha}$. This shows that n/d is divisible by every prime divisor of d. Therefore it follows from (1.7) and (1.8) that G has a fixed point free complex representation whose degree is 2^{α} if G is of type I with $d = 2^{\alpha}$, and 4 if G is of type II with $d = 2$. Thus if $\nu \geqq 3$, G acts freely and orthogonally on $S^{2^{\nu}-1}$. If $\nu = 2$, so does G of type I with $d = 2$. However (1.8) shows that G of type II with $d = 2$ can not act freely and orthogonally on S^{3}.

Case 2: $m = 1$, G is solvable.

In this case we have $d = 1$. Therefore it follows from (1.7) and (1.8) that G has a fixed point free complex representation whose degree is 1 if G is of type I, 2 if G is of type II, III or IV', and 4 if G is of type IV". Thus if $\nu \geqq 3$, G acts freely and orthogonally on $S^{2^\nu-1}$. If $\nu = 2$, so does G of type I, II, III or IV'. However (1.8) shows that G of type IV" can not act freely and orthogonally on S^3.

Case 3: $m = 1$, G is non-solvable.

For

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, p)$$

we have

$$X^i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \qquad (i = 0, 1, \cdots, p-1).$$

Therefore X generates a cyclic group of degree p. If we observe the order of G, it follows that this cyclic group is a p-Sylow subgroup of G. For

$$Y_i = \begin{pmatrix} \omega^i, & 0 \\ 0, & \omega^{-i} \end{pmatrix}, \qquad Z_i = \begin{pmatrix} 0 & -\omega^i \\ \omega^{-i} & 0 \end{pmatrix}$$

we have

$$Y_i X Y_i^{-1} = \begin{pmatrix} 1 & \omega^{2i} \\ 0 & 1 \end{pmatrix},$$

$$Z_i S X S^{-1} Z_i^{-1} = \begin{pmatrix} 1 & \omega^{2i+1} \\ 0 & 1 \end{pmatrix}.$$

Therefore it follows from (1.5) that 2^ν is a multiple of $p - 1$ if G is of type V, and that 2^ν is a multiple of $2(p-1)$ if G is of type VI. Thus G is of the following type V^*_α $(2 \leqq \alpha \leqq \nu)$ or VI^*_α $(2 \leqq \alpha \leqq \nu - 1)$.

V_α^*. $G = Z_n \times SL(2, p)$, where p is a prime of the form $2^\alpha + 1$, and $(n, p(p^2 - 1)) = 1$.

VI_α^*. G is generated by a group of type V_α^* and an element S satisfying the conditions in VI.

In particular, if $\nu = 2$, G is of type V_2^* and it acts freely and orthogonally on S^3 by (1.7) and (1.8). If $\nu = 3$, G is of type V_2^* or VI_2^*, and it acts freely and orthogonally on S^7 by (1.7) and (1.8). If $\nu = 4$, G is of type V_2^*, V_4^* or VI_2^*. The groups of type V_2^* or VI_2^* acts freely and orthogonally on S^{15}, but (1.7) shows that the groups of type V_4^* can not do so.

Remark 1. A prime of the form $2^\alpha + 1$ is called the *Fermat number*, and α is known to be of a power 2^β. But the converse is not true; for example $2^{32} + 1$ is divisible by 641.

Summing up the above arguments, we have proved the following two theorems.

(2.1) Theorem. *The conditions (A_3), (B_3), (C_3) and the following condition (D_3) are mutually equivalent for any finite group G.*

(D_3) *G is of type I with $d = 2^\alpha$ $(\alpha = 0, 1, 2)$, type II with $d = 2^\alpha$ $(\alpha = 0, 1)$, type III with $d = 1$, type IV with $d = 1$, type V with $d = 1$, or type VI with $d = 1$.*

(2.2) Theorem. *For $\nu \geqq 3$, the conditions (A_ν), (B_ν), (C_ν) and the following condition (D'_ν) are mutually equivalent for any finite solvable group G.*

(D'_ν) *G is of type I with $d = 2^\alpha$ $(0 \leqq \alpha < \nu)$, type II with*

$d = 2^{\alpha}$ $(\alpha = 0, 1)$, type III with $d = 1$, or type IV with $d = 1$.

For $\nu = 4$ we have also

(2.3) Theorem. The following two conditions for a finite group G are equivalent:

i) G satisfies the condition (C_4) but does not satisfy (A_4).

ii) G is of type V_4^*.

Proof. It has been proved in the arguments above that i) implies ii) and the groups of type V_4^* do not satisfy (A_4). It is easily seen that the groups of type V_4^* has only one element of order 2. Therefore it remains to prove that the groups of type V_4^* have period 16.

If $UXU^{-1} = X^i$ with $U \in SL(2, p)$, then it is easy to see that i is an even power of ω. Therefore it follows that the p-period of $SL(2, p)$ is $(p - 1)$. By (1.5) and (1.6), the 2-and 3-period of G divide 4. Since $|SL(2, 17)| = 2^5 \cdot 3^2 \cdot 17$, it holds that the period of $SL(2, 17)$ is 16. Thus we have the desired result, and the proof completes.

Here is a problem: Can the groups of type V_4^* act freely on S^{15}?

For $\nu = 2$ we have

(2.4) Theorem. The following two conditions for a finite group G are equivalent:

i) G satisfies the condition (C_2) but does not satisfy (A_2).

ii) G is of type II with $d = 2$ or type IV" with $d = 1$.

Proof. It has been proved that i) implies ii) and the groups of ii) do not satisfy (A_2).

Let G be of type II with $d = 2$, and we shall prove that G satisfies (C_2). It follows that $r \equiv -1\ (m)$ and

$$B^j A^i B^{-j} = A^{(-1)^j i}.$$

Therefore we have

$$(A^i B^j)^2 = A^{i(1+(-1)^j)} B^{2j},$$

$$(RA^i B^j)^2 = A^{i(\ell+(-1)^i)} B^{j(k+1)+n/2}$$

for any i, j. These show that if $A^i B^j$ is of order 2 then $i \equiv 0\ (m)$ and $j \equiv 0,\ n/2\ (n)$, and that $RA^i B^j$ is not of order 2. Thus G has only one element $B^{n/2}$ of order 2. Since the 2-Sylow subgroups of G are generalized quaternionic, the 2-period of G is 4. Let p be an odd prime dividing mn. If p divides m, $A^{m'}$ generates a p-Sylow subgroup of G, where $m = p^c m'$, $(m', p) = 1$. If p divides n, $B^{n'}$ generates a p-Sylow subgroup of G, where $n = p^c n'$, $(n', p) = 1$. It follows that

$$B^j A^{m'} B^{-j} = A^{(-1)^j m'}, \quad RB^j A^{m'} B^{-j} R^{-1} = A^{\pm m'},$$

$$A^i B^{n'} A^{-i} = B^{n'}, \quad RA^i B^{n'} A^{-i} R^{-1} = B^{\pm n'}.$$

Therefore we see that the p-period of G divides 4. Thus the period of G is 4.

Next, let G be of type IV" with $d = 1$. It is easy to see that G has only one element of order 2. Since the 2-Sylow subgroups of G are generalized quaternionic, the 2-period of G is 4. If p is an odd prime dividing n, then $B^{n'}$ generates a p-Sylow subgroup of G, where $n = p^c n'$, $(n', p) = 1$. If $p \neq 3$, we have $n' \equiv 0\ (3)$ and it follows that

$$PB^{n'}P^{-1} = B^{n'}, \ QB^{n'}Q^{-1} = B^{n'}, \ RB^{n'}R^{-1} = B^{\pm n'}.$$

Therefore we see that the p-period of G divides 4 if $p \neq 3$.

By (1.6) the same holds also for $p = 3$. Thus the period of G is 4. This completes the proof of (2.4).

Remark 2. If we use the notations in J. Milnor [3], it follows that the groups of type II with $d = 2$ are the products $Z_h \times Q(8g, s, t)$ with $(h, 2gst) = 1$, $s > t \geqq 1$, and the groups of type IV" with $d = 1$ are the products $Z_h \times P''_{48f}$ with f odd $\geqq 3$ and $(h, 6f) = 1$. In fact, B^{k+1}, $\{A, B^{(k-1)/2}, R\}$, $\{B^{(k-1)/2}, P, Q, R\}$ generate Z_h, $Q(8g, s, t)$, P''_{48f} respectively, where $h = (k - 1)/2$, $g = (k + 1)/4$, $f = (k + 1)/3$ and $0 < k < n$. Thus (2.4) is nothing but Theorem 3 of [3]. It is known that $Z_h \times Q(8g, s, t)$ for g even and $Z_h \times P''_{48f}$ with f not a power of 3 can not act freely on spheres of dimension $\equiv 3$ (8) (see [2], [4]). Here is a problem : Can the groups $Z_h \times Q(8g, s, t)$ with g odd and $P''_{48 \cdot 3^e}$ act freely on S^3?

3. Finite groups acting freely on S^{2p-1}

Let $Z_{q,p}$ be the metacyclic group with presentation $(X, Y \,;\, X^q = Y^p = 1,\ YXY^{-1} = X^{\sigma}\}$, where q is an odd integer, p a prime, $(\sigma - 1, q) = 1$, and σ is a primitive p^{th} root of 1 mod q.

By the arguments similar to § 2 but simpler, we have

(3.1) Theorem. Let p be an odd prime. Then the following two conditions for a finite group G are equivalent:

i) G has cohomology of period $2p$, has at most one element of degree 2, and can not act freely and orthogonally on S^{2p-1}.

ii) <u>G is of type $Z_h \times Z_{q,p}$ with $(h, pq) = 1$.</u>

Remark. It is known by T. Petrie [5] that $Z_{q,p}$ can act freely on S^{2p-1} if p is an odd prime. Here is a problem : If p is an odd prime and $h \neq 1$, can the groups $Z_h \times Z_{q,p}$ act freely on S^{2p-1}?

References

[1] H. Cartan and S. Eilenberg: Homological algebra, Princeton, 1956.

[2] R. Lee: Semicharacteristic classes, Topology 12 (1973), 183-199.

[3] J. Milnor: Groups which act on S^n without fixed points, Amer. J. Math. 79 (1957), 623-630.

[4] M. Nakaoka: Continuous maps of manifolds with involution I, Osaka J. Math. (to appear)

[5] T. Petrie: Free metacyclic group actions on homotopy spheres, Ann. of Math. 94 (1971), 108-124.

[6] M. Suzuki: On finite groups with cyclic Sylow subgroups for all odd primes, Amer. J. Math. 77 (1955), 657-691.

[7] R.G. Swan: The p-period of a finite group, Ill. J. Math. 4 (1960), 341-346.

[8] C.B. Thomas and C.T.C. Wall: The topological spherical space form problem I, Compositio Math. 23 (1971), 101-114.

[9] G. Vincent: Les groupes linéaires finis san points fixes, Comment. Math. Helv., 20 (1947), 117-171.

[10] J. Wolf: Spaces of constant curvature, McGraw-Hill, New-York (1967).

[11] H. Zassenhaus: Über endliche Fastkörper, Adh. Math. Semi. Univ. Hamburg, 11 (1936), 187-220.